ON THE IMAGE OF THE PERIOD MAP FOR POLARIZED HYPERKÄHLER MANIFOLDS

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Abstract. The moduli space for polarized hyperkähler manifolds of $\mathrm{K3}^{[m]}$-type or Kum_m-type with a given polarization type is not necessarily connected, which is a phenomenon that only happens for m large. The period map restricted to each connected component gives an open embedding into the period domain, and the complement of the image is a finite union of Heegner divisors. We give a simplified formula for the number of connected components, as well as a simplified criterion to enumerate the Heegner divisors in the complement. In particular, we show that the image of the period map may be different when restricted to different components of the moduli space.

1. Introduction

A *hyperkähler* manifold is a simply-connected compact Kähler manifold X such that $H^0(X,\Omega_X^2)=\mathbf{C}\omega$, where ω is a nowhere degenerate holomorphic 2-form on X. These manifolds form an important class among compact Kähler manifolds with trivial canonical bundle. For example, in dimension 2, these are precisely *K3 surfaces*. Higher-dimensional examples include manifolds of $\mathrm{K3}^{[m]}$-type (those deformation equivalent to Hilbert powers of K3 surfaces), Kum_m-type (generalized Kummer varieties and their deformations), and two families of examples, OG_6 and OG_{10}, discovered by O'Grady. These four families of examples are the only deformation types known up to now.

Given a hyperkähler manifold X, there is a quadratic form called the *Beauville–Bogomolov–Fujiki form* on the free abelian cohomology group $H^2(X,\mathbf{Z})$. It gives us a lattice structure of signature $(3,b_2-3)$ on the cohomology group and consequently, a polarized Hodge structure, which is fundamental in the study of hyperkähler manifolds. In the case of a K3 surface, this form coincides with the intersection product. If we fix the deformation type of X, the lattice is also fixed and will be denoted by Λ. We call an isometry $\eta\colon H^2(X,\mathbf{Z})\xrightarrow{\sim}\Lambda$ a *marking* of X. Denote by $\mathcal{M}_{\mathrm{marked}}$ the moduli space for marked hyperkähler manifolds (X,η) of the given deformation type. On each connected component $\mathcal{M}^0_{\mathrm{marked}}$ of the moduli space $\mathcal{M}_{\mathrm{marked}}$, the Hodge structures provide a *period map*

$$\wp^0_{\mathrm{marked}}\colon \mathcal{M}^0_{\mathrm{marked}}\longrightarrow \Omega_{\mathrm{marked}},$$

where

$$\Omega_{\mathrm{marked}}\coloneqq\{[x]\in\mathbf{P}(\Lambda_{\mathbf{C}})\mid (x,x)=0,(x,\bar{x})>0\}$$

is a complex manifold called the *period domain*. The global Torelli theorem, proven by Verbitsky, states that $\wp^0_{\mathrm{marked}}$ is surjective, generically injective, and identifies pairwise inseparable points.

On a projective hyperkähler manifold X, we may consider the extra datum of a *polarization*, that is, a primitive ample class $H\in H^2(X,\mathbf{Z})$. Any marking η maps H to a vector $\eta(H)\in\Lambda$,

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so it is reasonable to define the *polarization type* T of (X, H) as the $\mathrm{O}(\Lambda)$-orbit of $\eta(H)$ in Λ, which does not depend on the choice of the marking η. There is a quasi-projective moduli space $\mathcal{M}_T$ for polarized hyperkähler manifolds (X, H) of fixed polarization type T. For K3 surfaces, each polarization type T is uniquely determined by its square $2d$ and each moduli space $\mathcal{M}_{2d}$ is an irreducible quasi-projective variety of dimension 19. However, for their higher-dimensional analogues, the polarization types are more complicated to describe: apart from the square, there is another invariant, the *divisibility*. Moreover, Apostolov showed in [Apo14] that for some polarization types T on manifolds of $\mathrm{K3}^{[m]}$-type, the moduli space $\mathcal{M}_T$ may have several connected components. Onorati obtained similar results for Kum_m-type in [Ono16]. We shall review their results and give a simplified expression for the exact number of components in Section 3 (Proposition 3.4 and Proposition 3.5).

One can also consider the period map for polarized hyperkähler manifolds and its restriction to each connected component $\mathcal{M}_T^0$ of the polarized moduli space $\mathcal{M}_T$. We will use the letter τ to denote a *deformation type* of polarizations of type T. Such deformation types are in bijection with the connected components of $\mathcal{M}_T$, so we will write $\mathcal{M}_\tau$ instead of $\mathcal{M}_T^0$. In order to get rid of the choice of a marking, we consider the quotient of the corresponding period domain Ω, which is a hyperplane section inside Ω_{marked}, by the action of the elements in the orthogonal group $\mathrm{O}(\Lambda)$ that stabilize the polarization. In this way, we get a period domain $\mathcal{P}_T$, depending only on the polarization type T. But the global Torelli theorem no longer holds in this case, as the map from $\mathcal{M}_\tau$ to $\mathcal{P}_T$ might not be generically injective. In fact, it factors through $\mathcal{P}_\tau$, the quotient of Ω by a smaller group $\mathrm{Mon}(\Lambda)$, the *monodromy group*, which is a normal subgroup of $\mathrm{O}(\Lambda)$ for all the known deformation types (see Table 1). Thus the correct global Torelli theorem says that the polarized period map

$$\begin{array}{ccc} \wp_\tau\colon \mathcal{M}_\tau & \hookrightarrow & \mathcal{P}_\tau \\ & & \big\downarrow {\scriptstyle /G} \\ & & \mathcal{P}_T \end{array}$$

is an open immersion, where $\mathcal{P}_\tau$ is a covering space of $\mathcal{P}_T$ with finite deck transformation group G. The complement of the image of this open immersion is a finite union of divisors in $\mathcal{P}_\tau$. Intuitively, when the periods of the manifolds X in the family move towards the boundary of the image, the polarization H on X will move towards the boundary of the ample cone. Therefore, the determination of the divisors in the complement of the image is intimately related to the geometry of the ample cone for manifolds X in the family.

In the $\mathrm{K3}^{[m]}$-type case, the description of the ample cone was given by Bayer–Hassett–Tschinkel [BHT15], using the theory of Bayer–Macrì [BM14]. The description is based on a canonical embedding of $H^2(X, \mathbf{Z})$ into a larger lattice $\widetilde{\Lambda}$, known as the *Mukai lattice*. The ample cone can then be described using some numerical conditions. The analogous result for Kum_m-type was obtained by Yoshioka [Yos16]. We will review this in Section 4 and give a simplified description, without explicitly referring to the larger Mukai lattice (Proposition 4.5). We will use this description to characterize the divisors in the complement of the image of the period map. Note that the $\mathrm{K3}^{[2]}$-type case was completely treated in [DM19] (see also [Deb18, Appendix B]).

A natural question arises of whether for a given polarization type T, different connected components $\mathcal{M}_\tau$ of $\mathcal{M}_T$ have the same image in $\mathcal{P}_\tau$ under their corresponding period map. This question in general is not well-posed, as there is no canonical way to identify the period domains $\mathcal{P}_\tau$ for different components, due to the action of the deck transformation group G. Nevertheless, there is no problem of identification when G is trivial, and we provide a negative answer in the $\mathrm{K3}^{[m]}$-type case: by using our numerical description of the image, we construct in Section 5 an example where two connected components of the same $\mathcal{M}_T$ have different images in $\mathcal{P}_T$. We will also give another example where the group G is non-trivial and the image of the period map in $\mathcal{P}_\tau$ is not G-invariant above $\mathcal{P}_T$.

Notation. For a fixed deformation type of hyperkähler manifolds, we use $\mathcal{M}_{\text{marked}}$ (resp. $\mathcal{M}_T$) to denote the marked (resp. polarized) moduli space. The notation $\mathcal{M}^0$ will be used to denote a connected component of the corresponding moduli space $\mathcal{M}$.

For a positive integer n, we denote by $\rho(n)$ the number of distinct prime divisors of n and by $\widetilde{\rho}(n)$ the number $\rho(n)$ if n is odd and $\rho(n/2)$ if n is even. For a prime number p, we write $v_p(n)$ for the p-adic valuation of n.

To treat $\mathrm{K3}^{[m]}$-type and Kum_m-type manifolds simultaneously, we let $\widetilde{m} = m - 1$ for $\mathrm{K3}^{[m]}$-type and $\widetilde{m} = m + 1$ for Kum_m-type.

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2. Setup

In this section, we review the construction of the polarized period map and its relation with the monodromy group, following the work of Markman [Mar11, Section 4,7, and 8]. We reformulate some of the results to give a simpler presentation and to better suit our needs for later sections. We will consider a fixed deformation type of hyperkähler manifolds and denote by Λ the lattice defined by the Beauville–Bogomolov–Fujiki form on the second cohomology group, which has signature $(3, b_2 - 3)$.

First we recall the following definitions (*cf.* [Mar11, Definition 1.1]).

Definition 2.1. Let X and X' be hyperkähler manifolds of the given deformation type.

(i) An isomorphism $f\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(X',\mathbf{Z})$ is called a *parallel transport operator* if there exist a smooth and proper family $\pi\colon \mathcal{X} \to B$ of hyperkähler manifolds, with points $b, b' \in B$ and a path $\gamma\colon [0,1] \to B$ connecting b and b', such that $X \simeq \mathcal{X}_b$, $X' \simeq \mathcal{X}_{b'}$, and f is given as the parallel transport in the local system $R^2\pi_*\mathbf{Z}$ along γ.

(ii) An automorphism $f\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(X,\mathbf{Z})$ that is a parallel transport operator is called a *monodromy operator*. The subgroup of $\mathrm{O}(H^2(X,\mathbf{Z}))$ generated by monodromy operators is called the *monodromy group* of X and denoted by $\mathrm{Mon}(X)$.

(iii) If (X,H) and (X',H') are polarized hyperkähler manifolds, we define similarly a *polarized parallel transport operator* $f\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(X',\mathbf{Z})$ to be one induced by a path γ in a family of polarized hyperkähler manifolds. In other words, the local system $R^2\pi_*\mathbf{Z}$ admits a section h of ample classes, such that $h(b) = H$ and $h(b') = H'$.

In this paper, we will make the assumption that the monodromy group $\mathrm{Mon}(X)$ is a normal subgroup of $\mathrm{O}(H^2(X,\mathbf{Z}))$, in which case it can be identified as a subgroup $\mathrm{Mon}(\Lambda)$ of $\mathrm{O}(\Lambda)$. This holds for all known deformation types of hyperkähler manifolds.

A first property of the monodromy group $\mathrm{Mon}(\Lambda)$ can be given in terms of the *spinor norm*, which is the following homomorphism of groups

$$\sigma\colon \mathrm{O}(\Lambda_{\mathbf{R}}) \simeq \mathrm{O}(3, b_2-3) \longrightarrow \{\pm 1\},$$

given by the action on the orientation of a positive three-space W_3 of $\Lambda_{\mathbf{R}}$. In a more canonical way, we may consider the positive cone

$$\widetilde{\mathcal{C}}_\Lambda \coloneqq \{x \in \Lambda_{\mathbf{R}} \mid (x,x) > 0\}.$$

For any positive three-space W_3 in $\Lambda_{\mathbf{R}}$, $W_3 \smallsetminus \{0\}$ is a deformation retract of $\widetilde{\mathcal{C}}_\Lambda$. So an orientation of W_3 determines a generator of $H^2(W_3 \smallsetminus \{0\}, \mathbf{Z}) \simeq H^2(\widetilde{\mathcal{C}}_\Lambda, \mathbf{Z}) \simeq \mathbf{Z}$. The two generators of $H^2(\widetilde{\mathcal{C}}_\Lambda, \mathbf{Z})$ are called *orientation classes* of the positive cone $\widetilde{\mathcal{C}}_\Lambda$ and the spinor norm can be defined by the action on them (*cf.* [Mar11, Section 4]). For any subgroup G of $\mathrm{O}(\Lambda)$, we write G^+ for the subgroup of G consisting of elements of trivial spinor norm.

Proposition 2.2. *The monodromy group* $\mathrm{Mon}(\Lambda)$ *is contained in* $\mathrm{O}^+(\Lambda)$.

Proof. For a marked pair (X,η) with period $[x] \in \Omega_{\mathrm{marked}}$, we can take a Kähler class H on X and consider the orientation on the positive three-space $\mathbf{C}x \oplus \mathbf{R}\eta(H)$ given by the basis $\{\mathrm{Re}\, x, \mathrm{Im}\, x, \eta(H)\}$. This gives a distinguished orientation class of $\widetilde{\mathcal{C}}_\Lambda$, which is constant on each connected component $\mathcal{M}^0_{\mathrm{marked}}$ of the marked moduli space $\mathcal{M}_{\mathrm{marked}}$. Therefore every monodromy operator must have trivial spinor norm. □

From now on, we pick one connected component $\mathcal{M}^0_{\mathrm{marked}}$ of the marked moduli space $\mathcal{M}_{\mathrm{marked}}$. Recall from the introduction that we have the period map

$$\wp = \wp^0_{\mathrm{marked}}\colon \mathcal{M}^0_{\mathrm{marked}} \longrightarrow \Omega_{\mathrm{marked}}, \tag{1}$$

which is surjective by the global Torelli theorem. Let $h \in \Lambda$ be a primitive element of positive square. Consider the hyperplane section

$$\begin{aligned}\Omega_{\mathrm{marked}} \cap h^\perp =& \{[x] \in \Omega_{\mathrm{marked}} \mid (x,h) = 0\}\\ =& \{[x] \in \mathbf{P}(\Lambda_{\mathbf{C}}) \mid (x,x) = (x,h) = 0, (x,\bar{x}) > 0\}\end{aligned}$$

inside the marked period domain Ω_{marked}. It has two connected components denoted by Ω_h and Ω_{-h}. For any $[x] \in \Omega_h \sqcup \Omega_{-h}$, the real vector space $\mathbf{C}x \oplus \mathbf{R}h$ is a positive three-space in $\Lambda_{\mathbf{R}}$, but the orientation classes given by the basis $\{\mathrm{Re}\, x, \mathrm{Im}\, x, h\}$ are opposite on the two connected components. Since there is a distinguished orientation class for the connected component $\mathcal{M}^0_{\mathrm{marked}}$, up to interchanging Ω_h and Ω_{-h}, we may suppose that it coincides with $\{\mathrm{Re}\, x, \mathrm{Im}\, x, h\}$ for $[x] \in \Omega_h$ (and consequently it also coincides with $\{\mathrm{Re}\, x, \mathrm{Im}\, x, -h\}$ for $[x] \in \Omega_{-h}$).

Consider the preimages under the period map (1) of each of these two connected components. We denote them by $\mathcal{M}_h$ and $\mathcal{M}_{-h}$. Due to the surjectivity of the period map, both are non-empty divisors in $\mathcal{M}^0_{\mathrm{marked}}$. In fact, the union $\mathcal{M}_h \sqcup \mathcal{M}_{-h}$ is exactly the locus where the class $\eta^{-1}(h)$ is algebraic.

Proposition 2.3. *For a very general (X,η) in $\mathcal{M}_h$, the class $\eta^{-1}(h)$ is ample, while for a very general (X,η) in $\mathcal{M}_{-h}$, the class $\eta^{-1}(-h)$ is ample.*

Proof. For a very general element (X,η) in $\mathcal{M}_h$ with period $[x] \in \Omega_h$, the Néron–Severi group is generated by the class $H \coloneqq \eta^{-1}(h)$. In this case the Kähler cone coincides with the positive cone [Huy99, Corollary 7.2]. Since h is primitive of positive square, this implies that either H or $-H$ is ample. On the other hand, $[x]$ lies in Ω_h, so the orientation class given by $\{\operatorname{Re} x, \operatorname{Im} x, h\}$ coincides with the distinguished one, which can be given by $\{\operatorname{Re} x, \operatorname{Im} x, \eta(H')\}$ for some Kähler class H'. This implies that only H can be ample. By symmetry, we get the result for $-h$. ☐

By removing the locus inside $\mathcal{M}_h$ where $\eta^{-1}(h)$ is not ample, which is a countable union of closed complex analytic subsets, we get the following result [Mar11, Corollary 7.3].

Proposition 2.4 (Markman). *Let $\mathcal{M}_h^{\mathrm{amp}}$ be the locus in $\mathcal{M}_h$ that consists of marked pairs (X,η) such that $\eta^{-1}(h)$ is ample. Then $\mathcal{M}_h^{\mathrm{amp}}$ is connected and Hausdorff, and the marked period map $\wp$ restricts to an injective map from $\mathcal{M}_h^{\mathrm{amp}}$ onto a dense open subset of Ω_h (in the analytic topology).*

Remark 2.5. In Markman's survey, the domains Ω_h, $\mathcal{M}_h$, and $\mathcal{M}_h^{\mathrm{amp}}$ are denoted as $\Omega^+_{h^\perp}$, $\mathfrak{M}^+_{h^\perp}$, and $\mathfrak{M}^a_{h^\perp}$. We believe our notation is simpler and better reflects the symmetry between h and $-h$: we may identify $\Omega^-_{h^\perp} = \Omega^+_{(-h)^\perp}$ as Ω_{-h}, and $\mathfrak{M}^-_{h^\perp} = \mathfrak{M}^+_{(-h)^\perp}$ as $\mathcal{M}_{-h}$.

The connectedness of the locus $\mathcal{M}_h^{\mathrm{amp}}$ implies the following result [Mar11, Corollary 7.4], which determines whether two polarized hyperkähler manifolds lie in the same connected component of the polarized moduli space.

Proposition 2.6 (Markman). *A parallel transport operator*

$$f\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(X',\mathbf{Z})$$

is a polarized parallel transport operator from (X,H) to (X',H') if and only if $f(H)=H'$.

Definition 2.7. We fix one connected component $\mathcal{M}^0_{\mathrm{marked}}$ of the marked moduli space $\mathcal{M}_{\mathrm{marked}}$ as before. Given a polarized pair (X,H), choose a marking η such that (X,η) lies in $\mathcal{M}^0_{\mathrm{marked}}$. We define the *polarization type* T of (X,H) to be the $\mathrm{O}(\Lambda)$-orbit of $\eta(H)$ in Λ. We also denote by τ the $\mathrm{Mon}(\Lambda)$-orbit of $\eta(H)$ in Λ, which is contained in T. This orbit is clearly constant on each connected component $\mathcal{M}^0_T$ of $\mathcal{M}_T$, so we have a map

$$\{\text{connected components of } \mathcal{M}_T\} \longrightarrow \{\mathrm{Mon}(\Lambda)\text{-orbits contained in } T\}, \tag{2}$$

which may depend on the initial choice of the connected component $\mathcal{M}^0_{\mathrm{marked}}$. We will call the orbit τ the *deformation type* of (X,H).

Proposition 2.6 can be used to show that the deformation type defined here is the good notion. More precisely, we have the following result.

Proposition 2.8. *Let T be a polarization type, in other words, an $\mathrm{O}(\Lambda)$-orbit of a primitive element of positive square. The map* (2) *above gives a bijection from the set of connected components of $\mathcal{M}_T$ to the set of $\mathrm{Mon}(\Lambda)$-orbits contained in T.*

Proof. For the injectivity, suppose that two polarized pairs (X,H) and (X',H') have the same deformation type, which means that we may choose markings η and η' such that (X,η) and (X',η') both lie in the fixed connected component $\mathcal{M}^0_{\mathrm{marked}}$, and $\eta(H)$ and $\eta'(H')$ have the same $\mathrm{Mon}(\Lambda)$-orbit in Λ. We want to show that (X,H) and (X',H') lie in the same connected component of $\mathcal{M}_T$.

Suppose that there exists some $\phi \in \mathrm{Mon}(\Lambda)$ such that $\phi\circ\eta(H) = \eta'(H')$. By the definition of $\mathrm{Mon}(\Lambda)$, the marking $(X, \phi\circ\eta)$ is also in $\mathcal{M}^0_{\mathrm{marked}}$. The isomorphism $\eta'^{-1}\circ\phi\circ\eta$ is a parallel transport operator that takes H to H' so, by Proposition 2.6, it is a polarized one, that is, (X,H) and (X',H') are indeed connected by some path in the polarized moduli space $\mathcal{M}_T$.

For the surjectivity, since the locus $\mathcal{M}_h^{\mathrm{amp}}$ is non-empty for every $h \in T$, the class h can always be realized as the image $\eta(H)$ for some polarized pair (X, H) and a marking η with (X, η) lying in the fixed connected component $\mathcal{M}^0_{\mathrm{marked}}$. This in particular means that every $\mathrm{Mon}(\Lambda)$-orbit can be realized as the deformation type of some polarized pair. □

So for a given polarization type T, once we picked a connected component $\mathcal{M}^0_{\mathrm{marked}}$, we can distinguish each connected component $\mathcal{M}^0_T$ of $\mathcal{M}_T$ by its deformation type τ. We can thus write $\mathcal{M}_\tau$ instead of $\mathcal{M}^0_T$.

A first observation is that, if the group $\mathrm{Mon}(\Lambda)$ is a proper subgroup of $\mathrm{O}(\Lambda)$, an $\mathrm{O}(\Lambda)$-orbit may contain several $\mathrm{Mon}(\Lambda)$-orbits and consequently, the corresponding polarized moduli space $\mathcal{M}_T$ may have several components. As the result of Apostolov [Apo14] shows, this is indeed the case for certain polarization types of $\mathrm{K3}^{[m]}$-type manifolds. We will give a simplified expression for the exact number of components in Proposition 3.4.

Finally, we explain the construction of the polarized period map and the statement of the polarized global Torelli theorem, as mentioned in the introduction. For a polarization type T, we consider the connected component $\mathcal{M}^0_T = \mathcal{M}_\tau$ of the polarized moduli space $\mathcal{M}_T$ corresponding to a $\mathrm{Mon}(\Lambda)$-orbit τ and pick some $h \in \tau$. We consider the stabilizer groups

$$\mathrm{O}(\Lambda, h) \coloneqq \{\phi \in \mathrm{O}(\Lambda) \mid \phi(h) = h\} \quad \text{and} \quad \mathrm{Mon}(\Lambda, h) \coloneqq \mathrm{Mon}(\Lambda) \cap \mathrm{O}(\Lambda, h).$$

For a polarized pair (X, H) of deformation type τ, if we pick a suitable marking η in the connected component $\mathcal{M}^0_{\mathrm{marked}}$ such that $\eta(H) = h$ then, by the ampleness of the class H, the marked pair (X, η) must lie in $\mathcal{M}_h^{\mathrm{amp}}$. By quotienting out the action of the monodromy group, we get the following result [Mar11, Lemma 8.1, Lemma 8.3, and Theorem 8.4].

Theorem 2.9 (Markman)**.**

(i) *The marked period map* (1) *descends to an open embedding of analytic spaces*

$$\mathcal{M}_h^{\mathrm{amp}} / \mathrm{Mon}(\Lambda, h) \hookrightarrow \Omega_h / \mathrm{Mon}(\Lambda, h),$$

where the second quotient $\Omega_h / \mathrm{Mon}(\Lambda, h)$ is a normal quasi-projective variety by Baily–Borel theory. We denote this quotient by $\mathcal{P}_\tau$, since if we choose another $h' \in \tau$, the two quotients are canonically isomorphic.

(ii) *For each $h \in \tau$, there is an isomorphism of analytic spaces*

$$\mathcal{M}_\tau \xrightarrow{\sim} \mathcal{M}_h^{\mathrm{amp}} / \mathrm{Mon}(\Lambda, h).$$

The composition with the above embedding gives the polarized period map

$$\wp_\tau \colon \mathcal{M}_\tau \hookrightarrow \mathcal{P}_\tau,$$

which is an open immersion of algebraic varieties.

Notice that if τ and τ' are different $\mathrm{Mon}(\Lambda)$-orbits contained in T, the quotients $\mathcal{P}_\tau$ and $\mathcal{P}_{\tau'}$ are isomorphic but in general not canonically. This can be seen as follows. We consider the quotient $(\Omega_h \sqcup \Omega_{-h}) / \mathrm{O}(\Lambda, h) \simeq \Omega_h / \mathrm{O}^+(\Lambda, h)$, which is again a normal quasi-projective variety. This quotient can be denoted by $\mathcal{P}_T$, since if another $h' \in T$ is chosen, the two quotients are canonically isomorphic. We see that $\mathcal{P}_\tau$ is a covering space of $\mathcal{P}_T$ and it admits an action of the group $\mathrm{O}^+(\Lambda, h) / \mathrm{Mon}(\Lambda, h)$, not necessarily free. The deck transformation group G will be some quotient of this group. Thus we have a diagram

$$\begin{array}{rcl} \wp_\tau\colon \mathcal{M}_\tau \hookrightarrow \mathcal{P}_\tau & = & \Omega_h/\operatorname{Mon}(\Lambda,h) \\ \big\downarrow /G & & \\ \mathcal{P}_T & = & \Omega_h/\operatorname{O}^+(\Lambda,h) \end{array} \tag{3}$$

In particular, when G is non-trivial, for two deformation types τ and τ', there is no canonical isomorphism between the period domains $\mathcal{P}_\tau$ and $\mathcal{P}_{\tau'}$: any two such isomorphisms differ by the action of an element in G (to be more precise, in this case we have two groups G_τ and $G_{\tau'}$ that are non-canonically isomorphic).

Remark 2.10. For K3 surfaces, the monodromy group $\operatorname{Mon}(\Lambda)$ coincides with $\operatorname{O}^+(\Lambda)$, and each polarization is characterized by its square $2d$. Each period domain $\mathcal{P}_T = \mathcal{P}_{2d}$ is given above as the quotient $(\Omega_h \sqcup \Omega_{-h})/\operatorname{O}(\Lambda,h)$. This is usually formulated in terms of the orthogonal lattice $h^\perp$: the hyperplane section $(\Omega_h \sqcup \Omega_{-h})$ can be identified as the following space

$$\Omega_{h^\perp} \coloneqq \Big\{[x] \in \mathbf{P}\Big((h^\perp)_\mathbf{C}\Big) \;\Big|\; (x,x)=0, (x,\bar{x})>0\Big\},$$

and by Proposition 2.13 below, the group $\operatorname{O}(\Lambda,h)$ restricts to a subgroup $\widetilde{\operatorname{O}}(h^\perp)$ of $\operatorname{O}(h^\perp)$, so $\mathcal{P}_{2d}$ can also be given as the quotient $\Omega_{h^\perp}/\widetilde{\operatorname{O}}(h^\perp)$.

Remark 2.11. Another subtlety is that the polarized period map depends on the initial choice of the connected component $\mathcal{M}^0_{\text{marked}}$ for the definition of deformation types: if we choose another connected component by acting on the marking using an element in $\operatorname{Mon}(\Lambda)\cdot\operatorname{O}(\Lambda,h)$, the deformation type—the $\operatorname{Mon}(\Lambda)$-orbit—of $\mathcal{M}^0_T$ is still τ, but the period map is acted on by some element in G; if we choose another connected component by acting on the marking using an element in the larger group $\operatorname{O}(\Lambda)$, the deformation type of $\mathcal{M}^0_T$ may change to an entirely different τ', in which case the period map maps the component $\mathcal{M}^0_T$ to a different $\mathcal{P}_{\tau'}$ that, as we already stated, can only be identified with $\mathcal{P}_\tau$ up to the action of some element in G. In Markman's survey, this subtlety is handled by taking disjoint copies of $\mathcal{M}^{\text{amp}}_h$ (resp. Ω_h) and by quotienting out by the action of $\operatorname{O}(\Lambda)$ to get a canonically defined polarized moduli space (resp. polarized period domain). This approach is certainly more canonical as it does not depend on the particular choice of a connected component $\mathcal{M}^0_{\text{marked}}$. However, it is more difficult to describe the connected components of $\mathcal{M}_T$ in this setting.

Before ending this section, we review some lattice theoretical results that will be used later. We first recall some basic definitions. Let Λ be a lattice with isometry group $\operatorname{O}(\Lambda)$. The *divisibility* $\operatorname{div}(x)$ of a primitive element x in Λ is the positive generator γ of the subgroup (x,Λ) of $\mathbf{Z}$. The *discriminant group* of Λ is the finite abelian group $D(\Lambda) \coloneqq \Lambda^\vee/\Lambda$. We define $x_* \coloneqq [x/\operatorname{div}(x)]$, which is an element of $D(\Lambda)$ of order $\operatorname{div}(x)$. When Λ is even, the quadratic form on Λ induces a $(\mathbf{Q}/2\mathbf{Z})$-valued quadratic form on $D(\Lambda)$, and there is a natural homomorphism $\chi\colon \operatorname{O}(\Lambda) \to \operatorname{O}(D(\Lambda))$. In this case, we let $\widetilde{\operatorname{O}}(\Lambda)$ and $\widehat{\operatorname{O}}(\Lambda)$ be the respective preimages of $\{1\}$ and $\{\pm 1\}$ by χ. We have the following results from lattice theory.

Proposition 2.12 ([Nik79, Theorem 1.14.2])**.** *For any even indefinite lattice Λ of rank larger than or equal to the minimal number of generators of $D(\Lambda)$ plus 2, the homomorphism $\chi\colon \operatorname{O}(\Lambda) \to \operatorname{O}(D(\Lambda))$ is surjective.*

Proposition 2.13 ([GHS10, Lemma 3.2]). *Let Λ be an even unimodular lattice and let x be an element of Λ with non-zero square. Denote by $x^\perp$ the orthogonal of x in Λ. We have*

$$\mathrm{O}(\Lambda, x)|_{x^\perp} = \widetilde{\mathrm{O}}(x^\perp),$$

where $\mathrm{O}(\Lambda, x)$ is the stabilizer group of x in $\mathrm{O}(\Lambda)$.

Proposition 2.14 (Eichler's criterion, [GHS10, Lemma 3.5]). *Let Λ be an even lattice which contains at least two orthogonal copies of the hyperbolic plane U. The $\widetilde{\mathrm{O}}(\Lambda)$-orbit of a primitive element x is determined by its square x^2 and the class $x_* = [x/\operatorname{div}(x)]$ in $D(\Lambda)$.*

The Eichler's criterion can be slightly strengthened by replacing $\widetilde{\mathrm{O}}(\Lambda)$ with smaller subgroups.

Proposition 2.15. *Under the same assumption for Λ as above, for a primitive element x, the following three orbits coincide*

$$\widetilde{\mathrm{O}}(\Lambda)x = \widetilde{\mathrm{SO}}(\Lambda)x = \widetilde{\mathrm{SO}}^+(\Lambda)x.$$

In particular, all three orbits are determined by the square x^2 and the class x_* in $D(\Lambda)$.

Proof. Write $\Lambda = U_1 \oplus U_2 \oplus \Lambda_0$ where U_1 and U_2 are two copies of the hyperbolic plane U. Since U is unimodular, by Eichler's criterion, we may find $\phi \in \widetilde{\mathrm{O}}(\Lambda)$ such that $\phi(x) \in U_2 \oplus \Lambda_0$. Take $u, v \in U_1$ with $u^2 = 2$ and $v^2 = -2$, then the reflections R_u, R_v lie in $\mathrm{O}(\Lambda, \phi(x))$ and they satisfy $\sigma(R_u) = -1$, $\sigma(R_v) = 1$, $\chi(R_u) = \chi(R_v) = 1$, and $\det(R_u) = \det(R_v) = -1$.

Now for $\varphi \in \widetilde{\mathrm{O}}(\Lambda)$ with $\det(\varphi) = -1$, we have $\varphi(x) = \varphi \circ \phi^{-1} \circ \phi(x) = \varphi \circ \phi^{-1} \circ R_u \circ \phi(x)$, and the element $\varphi \circ \phi^{-1} \circ R_u \circ \phi$ has determinant 1, so $\varphi(x)$ lies in the same $\widetilde{\mathrm{SO}}(\Lambda)$-orbit as x and we get $\widetilde{\mathrm{O}}(\Lambda)x = \widetilde{\mathrm{SO}}(\Lambda)x$.

Similarly, for $\varphi \in \widetilde{\mathrm{SO}}(\Lambda)$ with $\sigma(\varphi) = -1$, we have $\varphi(x) = \varphi \circ \phi^{-1} \circ \phi(x) = \varphi \circ \phi^{-1} \circ R_u \circ R_v \circ \phi(x)$, and the element $\varphi \circ \phi^{-1} \circ R_u \circ R_v \circ \phi$ lies in $\widetilde{\mathrm{SO}}^+(\Lambda)$, so we get $\widetilde{\mathrm{SO}}(\Lambda)x = \widetilde{\mathrm{SO}}^+(\Lambda)x$. □

3. Monodromy group and number of components

In this section, we will calculate the number of components of the moduli space $\mathcal{M}_T$ of a given polarization type T, for all known deformation types. The polarization type determines the square and the divisibility of its elements, but the converse is in general not true: we will calculate the number of T with given square and divisibility.

First we recollect the descriptions for the lattice $\Lambda = H^2(X, \mathbf{Z})$ and the monodromy group $\mathrm{Mon}(\Lambda)$ for all known deformation types. The lattice structures for $\mathrm{K3}^{[m]}$ and Kum_m are known by Beauville [Bea83], and for OG_6 and OG_{10} they are computed by Rapagnetta [Rap08]. The monodromy group is computed by Markman in the $\mathrm{K3}^{[m]}$-case, Markman and Mongardi in the Kum_m-case [Mar22, Mon16], Mongardi–Rapagnetta for OG_6 [MR21], and Onorati for OG_{10} [Ono22].

Theorem 3.1. *The descriptions for the lattice $\Lambda = H^2(X, \mathbf{Z})$ and the monodromy group $\mathrm{Mon}(\Lambda)$ for all known deformation types are as follows.*

	$\Lambda = H^2(X, \mathbf{Z})$	$D(\Lambda)$	$\mathrm{Mon}(\Lambda)$
K3	$U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$	0	$\mathrm{O}^+(\Lambda)$
$\mathrm{K3}^{[m]}$	$\Lambda_{\mathrm{K3}} \oplus \langle -(2m-2) \rangle$	$\mathbf{Z}/(2m-2)\mathbf{Z}$	$\widehat{\mathrm{O}}^+(\Lambda)$
Kum_m	$U^{\oplus 3} \oplus \langle -(2m+2) \rangle$	$\mathbf{Z}/(2m+2)\mathbf{Z}$	$\left\{ g \in \widehat{\mathrm{O}}^+(\Lambda) \;\middle|\; \chi(g) \cdot \det(g) = 1 \right\}$
OG_6	$U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}$	$(\mathbf{Z}/2\mathbf{Z})^2$	$\mathrm{O}^+(\Lambda)$
OG_{10}	$\Lambda_{\mathrm{K3}} \oplus \begin{pmatrix} -6 & 3 \\ 3 & -2 \end{pmatrix}$	$\mathbf{Z}/3\mathbf{Z}$	$\mathrm{O}^+(\Lambda)$

TABLE 1. Lattice and monodromy group for known deformation types

Here U is the hyperbolic plane, $E_8(-1)$ is the E_8-lattice with negative definite form, and $\langle k \rangle$ is the lattice generated by one primitive element with square k.

We may compute the number of components for a given polarization type T using Proposition 2.8. We first prove a lemma concerning the orthogonal group of the discriminant group $D(\Lambda)$.

Lemma 3.2. *Let D be a cyclic group of order $2n$ with a quadratic form $q\colon D \to \mathbf{Q}/2\mathbf{Z}$. If there is a generator $g \in D$ with $q(g) = \frac{1}{2n}$, then*

$$\mathrm{O}(D) = \left\{ g \longmapsto ag \;\middle|\; \begin{array}{c} a \in \mathbf{Z}/2n\mathbf{Z} \\ a^2 \equiv 1 \pmod{4n} \end{array} \right\} \simeq (\mathbf{Z}/2\mathbf{Z})^{\rho(n)},$$

where $\rho(n)$ denotes the number of distinct prime divisors of n.

Proof. Write $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $r = \rho(n)$. If n is odd, a is determined by the conditions $a \equiv 1 \pmod 2$ and $a \equiv \pm 1 \pmod{p_i^{\alpha_i}}$; if n is even, we let $p_1 = 2$, then a is determined by the conditions $a \equiv \pm 1 \pmod{2^{\alpha_1+1}}$ and $a \equiv \pm 1 \pmod{p_i^{\alpha_i}}$ for $i \geq 2$. In both cases, we have $\mathrm{O}(D) \simeq (\mathbf{Z}/2\mathbf{Z})^{\rho(n)}$. □

The Eichler's criterion allows us to compute the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits.

Lemma 3.3. *Let Λ be an even lattice containing at least two orthogonal copies of the hyperbolic plane U, such that the discriminant group $D(\Lambda)$ is cyclic of order $2n$. Then for each $\mathrm{O}(\Lambda)$-orbit T of a primitive element with divisibility γ, the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits contained in T is equal to $2^{\widetilde{\rho}(\gamma)}$, where $\widetilde{\rho}(n)$ is equal to $\rho(n)$—the number of distinct prime divisors of n—if n is odd, and $\rho(n/2)$ if n is even.*

Proof. Fix one element $h \in T$ so that T is the set $\{\phi(h) \mid \phi \in \mathrm{O}(\Lambda)\}$. By Eichler's criterion (Proposition 2.14), as the square is fixed, the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits is the same as the number of possible values of $\big(\phi(h)\big)_* = \chi(\phi)(h_*) \in D(\Lambda)$ for all $\phi \in \mathrm{O}(\Lambda)$. The lattice Λ satisfies the condition in Proposition 2.12, so the homomorphism $\chi\colon \mathrm{O}(\Lambda) \to \mathrm{O}(D(\Lambda))$ is surjective. Therefore it suffices to count the number of possible $ah_* \in D(\Lambda)$ for all $a \in \mathrm{O}(D(\Lambda))$. Since h is primitive of divisibility γ, the class $h_* = [h/\gamma]$ is of order γ. Viewing the isometry a as an element of $\mathbf{Z}/2n\mathbf{Z}$, we therefore need to count the number of possible remainders of a modulo γ under the quotient map $\mathbf{Z}/2n\mathbf{Z} \to \mathbf{Z}/\gamma\mathbf{Z}$.

Using a similar argument as in the proof of Lemma 3.2, we write $\gamma = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $r = \rho(\gamma)$. If γ is odd, then a modulo γ can take all the values satisfying $a \equiv \pm 1 \pmod{p_i^{\alpha_i}}$. If γ is even, let $p_1 = 2$; if γ is not divisible by 4, that is, $\alpha_1 = 1$, then a modulo γ can take all the values satisfying $a \equiv 1 \pmod 2$ and $a \equiv \pm 1 \pmod{p_i^{\alpha_i}}$; if $\alpha_1 \geq 2$, a modulo γ can take all the values satisfying $a \equiv \pm 1 \pmod{2^{\alpha_1+1}}$ and $a \equiv \pm 1 \pmod{p_i^{\alpha_i}}$ for $i \geq 2$. Combining all three cases, the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits is equal to $2^{\widetilde{\rho}(\gamma)}$. $\square$

Now we can compute the number of connected components.

Proposition 3.4. *Let X be a hyperkähler manifold and T be a polarization type of divisibility γ on X.*

- *If X is of* $\mathrm{K3}^{[m]}$*-type or* Kum_m*-type, the number of connected components of the polarized moduli space $\mathcal{M}_T$ is equal to* $2^{\max(\widetilde{\rho}(\gamma)-1,0)}$.
- *If X is of* OG_6*-type or* OG_{10}*-type, the polarized moduli space $\mathcal{M}_T$ is connected.*

Proof. As Proposition 2.8 shows, the number of connected components of $\mathcal{M}_T$ is equal to the number of $\mathrm{Mon}(\Lambda)$-orbits contained in the $\mathrm{O}(\Lambda)$-orbit T. We fix one element $h \in T$, so T is the set $\{\phi(h) \mid \phi \in \mathrm{O}(\Lambda)\}$.

Case $\mathrm{K3}^{[m]}$**:** The discriminant group $D(\Lambda)$ is cyclic of order $2m-2$, so Lemma 3.3 applies and the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits contained in T is equal to $2^{\widetilde{\rho}(\gamma)}$.

Since the subgroup $\widehat{\mathrm{O}}(\Lambda)$ is generated by $\widetilde{\mathrm{O}}(\Lambda)$ and $-\mathrm{Id}$, we see that if h and $-h$ are in the same $\widetilde{\mathrm{O}}(\Lambda)$-orbit, that is, when γ is 1 or 2, the number of $\widehat{\mathrm{O}}(\Lambda)$-orbits is the same as the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits; otherwise it should be divided by 2. So this gives $2^{\max(\widetilde{\rho}(\gamma)-1,0)}$ as the number of $\widehat{\mathrm{O}}(\Lambda)$-orbits.

To conclude, we show that the $\widehat{\mathrm{O}}(\Lambda)$-orbits and the $\widehat{\mathrm{O}}^+(\Lambda)$-orbits are the same. Following the proof of Proposition 2.15, there is an element $R \in \mathrm{O}(\Lambda, h)$ (namely R_u) with $\sigma(R) = -1$ and $\chi(R) = 1$. Now for $\phi \in \widehat{\mathrm{O}}(\Lambda)$ with $\sigma(\phi) = -1$, we have $\phi(h) = \phi \circ R(h)$, where $\phi \circ R$ lies in $\widehat{\mathrm{O}}^+(\Lambda)$. So $\phi(h)$ lies in the same $\widehat{\mathrm{O}}^+(\Lambda)$-orbit as h and therefore $\widehat{\mathrm{O}}(\Lambda)h = \widehat{\mathrm{O}}^+(\Lambda)h$.

Case Kum_m**:** The discriminant group $D(\Lambda)$ is cyclic of order $2m+2$, so again Lemma 3.3 applies and we get $2^{\widetilde{\rho}(\gamma)}$ as the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits. By Proposition 2.15, this is also the number of $\widetilde{\mathrm{SO}}^+(\Lambda)$-orbits.

Moreover, following the proof of Proposition 2.15, there exists an element $R \in O(\Lambda, h)$ (namely $R_u \circ R_v$) such that $\sigma(R) = -1$, $\chi(R) = 1$, $\det(R) = 1$. On the other hand, we note that $\sigma(-\mathrm{Id}) = -1$, $\chi(-\mathrm{Id}) = -1$, $\det(-\mathrm{Id}) = -1$. This shows that $\mathrm{Mon}(\Lambda)$ is generated by $\widetilde{\mathrm{SO}}^+(\Lambda)$ and $-R$. If h and $-h = -R(h)$ are in the same $\widetilde{\mathrm{SO}}^+(\Lambda)$-orbit, that is, when γ is 1 or 2, then the number of $\mathrm{Mon}(\Lambda)$-orbits is the same as the number of $\widetilde{\mathrm{SO}}^+(\Lambda)$-orbits; otherwise it should be divided by 2. So again we obtain $2^{\max(\widetilde{\rho}(\gamma)-1,0)}$ as the number of $\mathrm{Mon}(\Lambda)$-orbits.

Case OG_6 **and** OG_{10}**:** In these two cases, the monodromy group is equal to $\mathrm{O}^+(\Lambda)$. Again, following the proof of Proposition 2.15, there exists a reflection $R \in O(\Lambda, h)$ (namely R_u) such that $\sigma(R) = -1$. So one may conclude that the $\mathrm{O}^+(\Lambda)$-orbit of h coincides with the entire $\mathrm{O}(\Lambda)$-orbit T. $\square$

We also have the following result on the number of polarization types with given square and divisibility on a hyperkähler manifold of $\mathrm{K3}^{[m]}$-type or Kum_m-type. Together with Proposition 3.4, this gives a refined version of Apostolov's [Apo14] result for $\mathrm{K3}^{[m]}$ and Onorati's [Ono16] result for Kum_m (*cf.* also [GHS10, Proposition 3.6]).

Proposition 3.5. *Let m, n, and γ be positive integers with $m \geq 2$. Let $\widetilde{m}$ be $m-1$ for the* $\mathrm{K3}^{[m]}$*-type and $m+1$ for the* Kum_m*-type, so in both cases we have $D(\Lambda) \simeq \mathbf{Z}/2\widetilde{m}\mathbf{Z}$. Moreover we assume that $\gamma \mid \gcd(2\widetilde{m}, 2n)$. For a prime divisor p of γ, set $\alpha \coloneqq \min(v_p(\widetilde{m}), v_p(n))$ and $\beta \coloneqq v_p(\gamma)$, where v_p is the p-adic valuation. Then there exists a polarization type T of square $2n$ and of divisibility γ, if and only if the following conditions are satisfied for all prime divisors p of γ:*

- *if $v_p(\widetilde{m}) \neq v_p(n)$, then $\beta \leq \alpha/2$;*
- *if $v_p(\widetilde{m}) = v_p(n) = \alpha$, then either $\beta \leq \alpha/2$, or $\beta > \alpha/2$ and $-n/\widetilde{m}$ is a square modulo $p^{2\beta-\alpha}$.*

The total number of these T is given by the product $\prod_{p \mid \gamma} N_p$, where for $p \geq 3$

$$N_p \coloneqq \begin{cases} \frac{1}{2}(p-1)p^{\beta-1} & \text{if } \beta \leq \alpha/2; \\ p^{\alpha-\beta} & \text{if } \beta > \alpha/2; \end{cases}$$

and for $p = 2$

$$N_2 \coloneqq \begin{cases} 1 & \text{if } \beta = 1; \\ 2^{\beta-2} & \text{if } \beta \geq 2, \beta \leq \alpha/2+1; \\ 2^{\alpha+1-\beta} & \text{if } \beta > \alpha/2+1. \end{cases}$$

Proof. For the $\mathrm{K3}^{[m]}$-type and the Kum_m-type, we have $\Lambda = \Lambda_0 \oplus \mathbf{Z}\delta$, where Λ_0 is an even unimodular lattice containing three orthogonal copies of the hyperbolic plane U, and δ is of square $-2\widetilde{m}$. The discriminant group is cyclic of order $2\widetilde{m}$, generated by δ_*.

We first study the existence of a polarization type with given square and divisibility. Let $h \in \Lambda$ be a primitive element of divisibility γ. If $\gamma = 1$, since Λ_0 contains orthogonal copies of U, it is clear that a polarization type of square $2n$ exists for all $n > 0$. So we will look at $\gamma \geq 2$. We write

$$h = \gamma a x + b\delta,$$

where $x \in \Lambda_0$ is primitive of square $x^2 = 2c$, with $a, b, c \in \mathbf{Z}$ such that $\gcd(\gamma a, 2\widetilde{m}) = \gamma$ and $\gcd(\gamma a, b) = 1$. Suppose that h is of square $2n$. We obtain the relation

$$2n = h^2 = 2ca^2\gamma^2 - b^2 \cdot 2\widetilde{m}.$$

For such an h to exist, it is necessary and sufficient that there exist some integer b satisfying

$$\gamma^2 \mid b^2\widetilde{m} + n.$$

For each prime divisor p of γ, since $\gcd(\gamma, b) = 1$, we see that b is not divisible by p. So if $v_p(\widetilde{m}) \neq v_p(n)$, then $v_p(b^2\widetilde{m} + n) = \min(v_p(\widetilde{m}), v_p(n))$ and we obtain the first condition; if $v_p(\widetilde{m}) = v_p(n) = \alpha$, then for $p^{2\beta} \mid b^2\widetilde{m} + n$ to hold we obtain the second condition.

Given the square and the divisibility, to count the number of such $\mathrm{O}(\Lambda)$-orbits T, we first count the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits. Any such element h can again be expressed as $\gamma a x + b\delta$. By Eichler's criterion, since the square is fixed, the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits is just the number of possible $h_* = \frac{b \cdot 2\widetilde{m}}{\gamma}\delta_*$, or equivalently, the number of possible remainders of b modulo γ. We thus express this number as the product $\prod_{p \mid \gamma} M_p$, where M_p is the number of possible remainders of b modulo p^β.

For $p \geq 3$, if $\beta \leq \alpha/2$, then we only need $\gcd(b, p) = 1$, thus M_p is equal to $(p-1)p^{\beta-1}$; if $\beta > \alpha/2$, then the equation $b^2 \equiv -n/\widetilde{m} \pmod{p^{2\beta-\alpha}}$ has two solutions, thus M_p is equal to $2p^{\alpha-\beta}$.

For $p=2$, as $\gcd(b,p)=1$, we see first that b is necessarily odd. If $\beta \le \alpha/2+1$, we will show that this is also sufficient, so M_2 is equal to $2^{\beta-1}$. To prove this, we distinguish three cases: if $\beta \le \alpha/2$, it is clear that $b^2\widetilde{m}+n$ is a multiple of $2^{2\beta}$; if $\beta = \alpha/2+1/2$, then $v_p(\widetilde{m}) = v_p(n) = \alpha$ and $b^2\widetilde{m}+n$ is a multiple of $2^{\alpha+1} = 2^{2\beta}$; if $\beta = \alpha/2+1$, then $v_p(\widetilde{m}) = v_p(n) = \alpha$ and $-n/\widetilde{m} \equiv 1 \pmod 4$, so $b^2\widetilde{m}+n$ is a multiple of $2^{\alpha+2} = 2^{2\beta}$. If $\beta > \alpha/2+1$, the equation $b^2 \equiv 1 \pmod{2^{2\beta-\alpha}}$ has two solutions modulo $2^{2\beta-\alpha-1}$, so M_2 is equal to $2 \times 2^{\alpha+1-\beta}$.

To conclude, as Lemma 3.3 shows that each $\mathrm{O}(\Lambda)$-orbit T contains $2^{\widetilde{\rho}(\gamma)}$ different $\widetilde{\mathrm{O}}(\Lambda)$-orbits, the number of T is given by $\prod_{p|\gamma} M_P$ divided by $2^{\widetilde{\rho}(\gamma)}$. We let $N_p = M_p/2$ for $p \ge 3$, $N_2 = M_2/2$ if $v_2(\gamma) \ge 2$, and $N_2 = M_2 = 1$ if $v_2(\gamma) = 1$. This gives the desired formula. □

For completeness, we also provide the results for OG_6 and OG_{10}.

Proposition 3.6. *Let n and γ be positive integers. For the OG_6-type and the OG_{10}-type, a polarization type T is uniquely determined by its square $2n$ and divisibility γ.*

- *For the OG_6-type, such T exists if and only if $\gamma = 1$, or $\gamma = 2$ and $n \equiv 2, 3 \pmod 4$;*
- *for the OG_{10}-type, such T exists if and only if $\gamma = 1$, or $\gamma = 3$ and $n \equiv 6 \pmod 9$.*

Proof. In both cases, since the lattice Λ contains orthogonal copies of U, the existence of a polarization type of square $2n$ and divisibility 1 is clear, and the uniqueness follows from Eichler's criterion.

For the OG_6-type, we write u and v for the two generators with square -2 so $\Lambda = \Lambda_0 \oplus \mathbf{Z}u \oplus \mathbf{Z}v$. Each primitive element h of divisibility 2 can be written as

$$h = 2ax + bu + cv,$$

where $x \in \Lambda_0$ is primitive with $x^2 = 2d$ and $a, b, c, d \in \mathbf{Z}$, such that $\gcd(2a, b, c) = 1$. In particular, b and c cannot be both even, and the class h_* is given by $(\bar{b}, \bar{c}) \in (\mathbf{Z}/2\mathbf{Z})^2$. Suppose that h is of square $2n$. We obtain the relation

$$2n = h^2 = 8a^2d - 2b^2 - 2c^2,$$

and we may deduce that $4 \mid n + b^2 + c^2$. If $n \not\equiv 2, 3 \pmod 4$ there are no integer solutions. If $n \equiv 2 \pmod 4$, then b and c must both be odd, so $h_* = (\bar{1}, \bar{1})$ and by Eichler's criterion all such h lie in the same $\widetilde{\mathrm{O}}(\Lambda)$-orbit, so the $\mathrm{O}(\Lambda)$-orbit is also unique. If $n \equiv 3 \pmod 4$, then b and c must be one odd one even, so h_* can either be $(\bar{1}, \bar{0})$ or $(\bar{0}, \bar{1})$, and by Eichler's criterion there are two $\widetilde{\mathrm{O}}(\Lambda)$-orbits, but the map that interchanges the coordinates u and v is an isometry, so these two lie in the same $\mathrm{O}(\Lambda)$-orbit, and again we get the uniqueness.

For the OG_{10}-type, we similarly write u and v for the two generators with matrix $\left(\begin{smallmatrix} -6 & 3 \\ 3 & -2 \end{smallmatrix}\right)$, so $\Lambda = \Lambda_0 \oplus \mathbf{Z}u \oplus \mathbf{Z}v$. Each primitive element h of divisibility 3 can be written as

$$h = 3ax + bu + 3cv,$$

where $x \in \Lambda_0$ is primitive with $x^2 = 2d$ and $a, b, c, d \in \mathbf{Z}$, such that $\gcd(3a, b, 3c) = 1$. In particular, b is not divisible by 3, and the class h_* is given by $\bar{b} \in \mathbf{Z}/3\mathbf{Z}$. Suppose that h is of square $2n$. We obtain the relation

$$2n = h^2 = 18a^d - 6b^2 + 18bc - 18c^2,$$

and we may deduce that $9 \mid n + 3b^2$, so we must have $n \equiv 6 \pmod 9$. By Eichler's criterion there are two $\widetilde{\mathrm{O}}(\Lambda)$-orbits depending on the value $h_* \in D(\Lambda) = \mathbf{Z}/3\mathbf{Z}$, but $-\mathrm{Id}$ interchanges the two non-zero classes in $D(\Lambda)$ so again the two lie in the same $\mathrm{O}(\Lambda)$-orbit. □

4. Image of the period map

We will now study the image of the polarized period map. For all known deformation types, the complement of the image in the period domain can be shown to be a finite union of divisors: we will give explicit numerical conditions describing these divisors. The image of the period map is closely related to the determination of the ample cone, which has been settled for all known deformation types, so we first review the results.

Recall that on $H^{1,1}(X,\mathbf{R})$ the Beauville–Bogomolov–Fujiki form induces a quadratic form of signature $(1,b_2-3)$, so the cone of positive classes has two connected components, and we call the one containing a Kähler class the *positive cone* and denote it by $\mathcal{C}_X$. The cone of all Kähler classes sits inside $\mathcal{C}_X$ and is denoted by $\mathcal{K}_X$. We also consider the *birational Kähler cone* $\mathcal{BK}_X$, which is the union $\bigcup f^{-1}\mathcal{K}_{X'}$ over all birational maps f from X to some other hyperkähler manifold X'. The Néron–Severi group $\mathrm{NS}(X)$ is a sublattice $H^2(X,\mathbf{Z})\cap H^{1,1}(X,\mathbf{R})$ inside $H^2(X,\mathbf{Z})$.

We have the following crucial notion: a divisor D on X is called a *wall divisor*, if $D^2<0$ and $f(D^\perp)\cap\mathcal{BK}_X=\varnothing$ for all monodromy operators f (*cf.* [Mon15, Definition 1.2] and [AV15, Definition 1.13]). The property of being a wall divisor is stable under parallel transport operators [Mon15, Theorem 3.1].

Theorem 4.1 (Mongardi)**.** *Let (X,η) and (X',η') be two marked hyperkähler manifolds lying in the same connected component $\mathcal{M}^0_{\text{marked}}$ of the marked moduli space. Let $D\in\mathrm{NS}(X)$ and $D'\in\mathrm{NS}(X')$ be divisors such that $\eta^{-1}\circ\eta(D)=D'$. Then D is a wall divisor on X if and only if D' is a wall divisor on X'.*

Once we picked a connected component $\mathcal{M}^0_{\text{marked}}$, we may extend this notion to elements of the lattice Λ: a class $\kappa\in\Lambda$ with $\kappa^2<0$ is called a *wall class*, if for all $(X,\eta)\in\mathcal{M}^0_{\text{marked}}$ such that the class $\eta^{-1}(\kappa)$ is of type $(1,1)$, it gives a wall divisor on X. Clearly the property only depends on the $\mathrm{Mon}(\Lambda)$-orbit of κ. Wall divisors give a chamber decomposition on the positive cone $\mathcal{C}_X$, and the Kähler cone $\mathcal{K}_X$ is given by one of the chambers.

For $\mathrm{K3}^{[m]}$-type and Kum_m-type, a numerical characterization for wall divisors is known. Write as before $\widetilde{m}=m-1$ for $\mathrm{K3}^{[m]}$-type and $\widetilde{m}=m+1$ for Kum_m-type. Recall that in these two cases, the lattice Λ has the form $\Lambda=\Lambda_0\oplus\mathbf{Z}\delta$, where Λ_0 is an even unimodular lattice containing three orthogonal copies of U, and δ is of square $-2\widetilde{m}$. We also consider the *Mukai lattice*

$$\widetilde{\Lambda}\coloneqq\Lambda_0\oplus U,$$

which is even and unimodular. For any vector $v\in\widetilde{\Lambda}$ of square $2\widetilde{m}$, the sublattice $v^\perp$ is isomorphic to Λ. Since all such v are in the same $\mathrm{O}(\widetilde{\Lambda})$-orbit due to the unimodularity of $\widetilde{\Lambda}$, we may fix $v=u_1+\widetilde{m}u_2$, where $\langle u_1,u_2\rangle$ is a copy of the hyperbolic plane U, and identify Λ as the sublattice $v^\perp$. In particular we set $\delta=u_1-\widetilde{m}u_2$.

When X is of $\mathrm{K3}^{[m]}$-type or Kum_m-type, there is an embedding of $H^2(X,\mathbf{Z})$ into $\widetilde{\Lambda}$, canonical up to the action of $\mathrm{O}(\widetilde{\Lambda})$ (see [Mar11, Corollary 9.5] for $\mathrm{K3}^{[m]}$-type, and [Wie18, Theorem 4.9] for Kum_m-type). For any such embedding, the orthogonal of its image is generated by a vector of square $2\widetilde{m}$. So we can assume that the image is exactly Λ, by mapping one of these generators to the fixed v using some element in $\mathrm{O}(\widetilde{\Lambda})$. In this way, we get a distinguished marking $\eta\colon H^2(X,\mathbf{Z})\xrightarrow{\sim}\Lambda$, canonical up to the action of $\{\pm\mathrm{Id}\}\cdot\mathrm{O}(\widetilde{\Lambda},v)|_\Lambda$. By Proposition 2.13, this group is equal to $\{\pm\mathrm{Id}\}\cdot\widetilde{\mathrm{O}}(\Lambda)=\widehat{\mathrm{O}}(\Lambda)$. Therefore we get the following result.

Proposition 4.2. *Let X be a hyperkähler manifold of* $\mathrm{K3}^{[m]}$*-type or* Kum_m*-type. There is a distinguished marking*

$$\eta\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} \Lambda \subset \widetilde{\Lambda},$$

canonical up to the action of $\widehat{\mathrm{O}}(\Lambda)$*. It induces an isometry between the two discriminant groups* $D(H^2(X,\mathbf{Z}))$ *and* $D(\Lambda) \simeq \mathbf{Z}/2\widetilde{m}\mathbf{Z}$*, canonical up to a sign. In other words, there is a canonical choice of a pair of generators* $\pm g$ *for* $D(H^2(X,\mathbf{Z}))$*, mapped to* $\pm\delta_*$ *under the isometry.*

Any monodromy operator must respect the choice of the pair of generators $\pm g$, so the monodromy group $\mathrm{Mon}(\Lambda)$ must lie in the subgroup $\widehat{\mathrm{O}}(\Lambda)$, which is indeed the case.

We now give the description of the Kähler cone $\mathcal{K}_X$ for these two cases. The $\mathrm{K3}^{[m]}$-case is due to the results of Bayer–Macrì, Bayer–Hassett–Tschinkel, and Mongardi (note that in [BHT15], the manifold X is assumed to be projective; this assumption can be removed using [Mon15] or [AV15, Theorem 1.17 and 1.19]). The Kum_m-case is due to Yoshioka [Yos16] (see also [Mon16]).

Theorem 4.3 (Bayer–Macrì, Bayer–Hassett–Tschinkel, Mongardi; Yoshioka)**.** *Let X be a hyperkähler manifold of* $\mathrm{K3}^{[m]}$*-type or* Kum_m*-type. Under the embedding*

$$\eta\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} \Lambda \hookrightarrow \widetilde{\Lambda},$$

we denote by $\widetilde{\Lambda}_{\mathrm{alg}}$ *the saturation of* $\eta(\mathrm{NS}(X)) \oplus \mathbf{Z}v$*. Consider the set*

$$S \coloneqq \begin{cases} \left\{ s \in \widetilde{\Lambda} \;\middle|\; s^2 \geq -2,\ |(s,v)| \leq \widetilde{m} = m-1 \right\} \smallsetminus \{0\} & \text{if } X \text{ is of } \mathrm{K3}^{[m]}\text{-type;} \\ \left\{ s \in \widetilde{\Lambda} \;\middle|\; s^2 \geq 0,\ 0 < |(s,v)| \leq \widetilde{m} = m+1 \right\} & \text{if } X \text{ is of } \mathrm{Kum}_m\text{-type.} \end{cases}$$

Then the Kähler cone $\mathcal{K}_X$ *is one of the connected components of the positive cone* $\mathcal{C}_X$ *cut out by the hyperplanes* $s^\perp$ *in* $\mathrm{NS}(X)_{\mathbf{R}}$*, for all* $s \in S \cap \widetilde{\Lambda}_{\mathrm{alg}}$*.*

Note that the particular choice of the embedding η does not matter here: because η is unique up to the action of $\mathrm{O}(\widetilde{\Lambda})$, and the set S is clearly $\mathrm{O}(\widetilde{\Lambda})$-invariant.

This description depends on the larger lattice $\widetilde{\Lambda}$, which is inconvenient to work with. Note that each $s \in S$ together with v span a rank-2 sublattice of $\widetilde{\Lambda}$, so we may consider its intersection with Λ, which is of rank 1, and pick a generator $\kappa \in \Lambda$. The hyperplane $s^\perp$ can then also be expressed as $\kappa^\perp$. Since the class κ lies in $\mathrm{NS}(X)$ if and only if s lies in $\widetilde{\Lambda}_{\mathrm{alg}}$, we may conclude that all wall classes arise this way from some $s \in S$. We now give a lattice theoretical result, which will yield a numerical criterion for wall classes $\kappa \in \Lambda$ that is intrinsic to the smaller lattice Λ.

Proposition 4.4. *Let* $\widetilde{\Lambda}$ *be a lattice of the form* $\Lambda_0 \oplus U$*, where* Λ_0 *is an even unimodular lattice and U is the hyperbolic plane with basis* u_1, u_2*. Let* $v = u_1 + \widetilde{m}u_2$ *and* $\delta = u_1 - \widetilde{m}u_2$*, and let Λ be the sublattice* $v^\perp = \Lambda_0 \oplus \mathbf{Z}\delta$*. Let* $\kappa \in \Lambda$ *be a primitive vector and write* $\kappa^2 = 2l$ *and* $\kappa_* = k\delta_* \in D(\Lambda) \simeq \mathbf{Z}/2\widetilde{m}\mathbf{Z}$*, where* $|k| \leq \widetilde{m}$*. Set* $d \coloneqq \gcd(2\widetilde{m}, k)$*.*

(i) *There is a unique integer c such that*

$$l = c\left(\frac{2\widetilde{m}}{d}\right)^2 - \widetilde{m}\left(\frac{k}{d}\right)^2.$$

(ii) *Let $a \in \mathbf{Z}_{\geq 0}$ be a non-negative integer. There is a non-zero element $s \in \widetilde{\Lambda}$ contained in the saturation of the sublattice generated by κ and v, such that*

$$s^2 \geq -2a, \quad |(s,v)| \leq \widetilde{m},$$

if and only if the integer c in (i) satisfies $c \geq -a$. When this is the case, there is one such element s with $s^2 = 2c$ and $(s,v) = -k$.

Proof. First we may assume that $k \geq 0$ by changing κ to $-\kappa$ if needed. Since $\kappa_* = [\kappa/\operatorname{div}(\kappa)]$ is equal to $k\delta_* = [k\delta/2\widetilde{m}]$ in $D(\Lambda)$, we may write

$$\frac{\kappa}{\operatorname{div}(\kappa)} = x + b\delta + \frac{k\delta}{2\widetilde{m}},$$

where $x \in \Lambda_0$ and $b \in \mathbf{Z}$. Since κ is integral and primitive, we see that $\operatorname{div}(\kappa) = \frac{2\widetilde{m}}{d}$. Now we let

$$s := \frac{d\kappa - kv}{2\widetilde{m}} = x + b\delta - ku_2,$$

which is an integral class in $\widetilde{\Lambda} \smallsetminus \{0\}$, with $|(s,v)| = |-k| \leq \widetilde{m}$. Let $s^2 = 2c$. We can easily verify that c is the integer satisfying the equality in (i). Moreover, if $c \geq -a$, the vector s provides the element we need in (ii).

Conversely, suppose that there is some other vector s' satisfying the condition in (ii), then we will show that $c \geq -a$ so the vector s itself satisfies the condition. We let $s'^2 = 2c'$ with $c' \geq -a$, and $(s',v) = -k'$ with $|k'| \leq \widetilde{m}$. Since $2\widetilde{m}s'$ lies in the direct sum $\mathbf{Z}\kappa \oplus \mathbf{Z}v$, there exists a unique integer d' such that

$$2\widetilde{m}s' = d'\kappa - k'v \quad \text{or equivalently} \quad s' = \frac{d'\kappa - k'v}{2\widetilde{m}}.$$

As κ is of divisibility $\frac{2\widetilde{m}}{d}$ in Λ, there is some $y \in \Lambda$ such that $(\kappa, y) = \frac{2\widetilde{m}}{d}$. We then have $(s', y) = \frac{d'}{d}$, so d divides d'. Set $d' = \lambda d$. We must have $\lambda \neq 0$: otherwise, s' is equal to $-\frac{k'}{2\widetilde{m}}v$; but $|k'| \leq \widetilde{m}$, so s' can only be 0, which contradicts our hypothesis. By changing s' to $-s'$ if needed, we may suppose that $\lambda \geq 1$. Then by looking at the integral class $s' - \lambda s$, we have $k' \equiv \lambda k \pmod{2\widetilde{m}}$. Write $k' = \lambda k - \mu \cdot 2\widetilde{m}$. Since $k' \leq \widetilde{m}$, we must have $\mu \geq 0$. Then we have $s' = \lambda s + \mu v$ and thus

$$\begin{aligned} s'^2 &= \lambda^2 s^2 + 2\lambda\mu(s,v) + \mu^2 v^2 \\ &= \lambda^2 s^2 - \mu(2\lambda k - \mu \cdot 2\widetilde{m})) \\ &= \lambda^2 s^2 - \mu(2k' + \mu \cdot 2\widetilde{m}) \leq \lambda^2 s^2, \end{aligned}$$

where the last inequality is due to $k' \geq -\widetilde{m}$ and $\mu \geq 0$. So we get $-a \leq c' \leq \lambda^2 c$ for some $\lambda \geq 1$, and we may conclude that $c \geq -a$. □

Proposition 4.5. *Let X be a hyperkähler manifold of* $\mathrm{K3}^{[m]}$*-type or* Kum_m*-type. Let g be one of the canonical generators of $D(H^2(X,\mathbf{Z}))$. The Kähler cone $\mathcal{K}_X$ is one of the components of the positive cone cut out by the hyperplanes $\kappa^\perp$, for all classes $\kappa \in \mathrm{NS}(X)$ satisfying the following numerical condition: writing $\kappa^2 = 2l$, $\kappa_* = kg$ with $0 \leq k \leq \widetilde{m}$, and $d = \gcd(2\widetilde{m}, k)$, then*

(4)
$$\begin{cases} l = c\left(\frac{2m-2}{d}\right)^2 - (m-1)\left(\frac{k}{d}\right)^2 & \text{for an integer } -1 \leq c < \frac{k^2}{4(m-1)} \quad \text{if } X \text{ is of } \mathrm{K3}^{[m]}\text{-type;} \\ l = c\left(\frac{2m+2}{d}\right)^2 - (m+1)\left(\frac{k}{d}\right)^2 & \text{for an integer } 0 \leq c < \frac{k^2}{4(m+1)} \quad \text{if } X \text{ is of } \mathrm{Kum}_m\text{-type.} \end{cases}$$

Proof. The $\mathrm{K3}^{[m]}$-case is obtained by combining Theorem 4.3 and Proposition 4.4 for $a = 1$, and the upper bound for c comes from $\kappa^2 = 2l < 0$. For the Kum_m-case, we use $a = 0$ and we note that k cannot take the value 0 because l needs to be negative. So we will only consider κ with $1 \le k \le \widetilde{m} = m+1$, and for such κ we indeed get an element s with $s^2 \ge 0$ and $0 < |(s,v)| = |-k| \le \widetilde{m} = m+1$. □

Remark 4.6.

- To enumerate all the wall divisors, we let k run from 0 to $\widetilde{m}$ and for each k, we let c run from -1 or 0 to $\left\lceil \frac{k^2}{4\widetilde{m}} \right\rceil - 1$ to get the corresponding l.
- As an example, for $\mathrm{K3}^{[2]}$-type, the pair (k,l) has three possibilities: $(0,-1)$, $(1,-5)$, and $(1,-1)$. Thus we get $\kappa^2 = -2$ and $\operatorname{div}(\kappa) = 1, 2$, or $\kappa^2 = -10$ and $\operatorname{div}(\kappa) = 2$. This was first conjectured in [HT09b]. See also [Mon15], where the cases of $\mathrm{K3}^{[m]}$-type for $m \le 4$ are worked out; and [HT09a], where some examples for Kum_m-type are given.
- Analogous results for OG_6 and OG_{10} are also established: wall divisors on a hyperkähler manifold X of OG_6-type are given by elements $\kappa \in \mathrm{NS}(X)$ with $\kappa^2 = -2$, or $\kappa^2 = -4$ and $\operatorname{div}(\kappa) = 2$ [MR21]; wall divisors on a hyperkähler manifold X of OG_{10}-type are given by elements $\kappa \in \mathrm{NS}(X)$ with $0 > \kappa^2 \ge -4$, or $0 > \kappa^2 \ge -24$ and $\operatorname{div}(\kappa) = 3$ [MO22].
- In particular, the Kawamata–Morrison conjecture holds for all known deformation types of hyperkähler manifolds by a result of Amerik–Verbitsky [AV15, Theorem 1.21]: for a given deformation type, since the square of a wall class is bounded below, the automorphism group $\mathrm{Aut}(X)$ acts on the set of faces of $\mathcal{K}_X$ with finitely many orbits.

We will now describe the image of the period map. Let τ be the deformation type of a polarization, and take an element $h \in \tau$. For a vector $u \in \Lambda$ with negative square and linearly independent of h, the hyperplane $u^\perp \subset \mathbf{P}(\Lambda_{\mathbf{C}})$ cuts a hyperplane section in the subset Ω_h and induces a divisor $\mathcal{H}_u$ in the period domain $\mathcal{P}_\tau = \Omega_h / \mathrm{Mon}(\Lambda, h)$ which is called a *Heegner divisor*. By abuse of notation, its image in $\mathcal{P}_T$ will also be denoted as $\mathcal{H}_u$.

Proposition 4.7. *Take a deformation type of hyperkähler manifolds for which the Kawamata–Morrison conjecture holds. Let τ be the deformation type of a polarization and take $h \in \tau$. The complement of the image of the period map $\wp_\tau$ in $\mathcal{P}_\tau$ is the union of the Heegner divisors $\mathcal{H}_\kappa$ induced by wall classes $\kappa \in \Lambda$ that are orthogonal to h.*

Proof. For $x \in \kappa^\perp$, if there is a polarized pair (X, H) of deformation type τ such that $\wp(X) = [x] \in \Omega_{\mathrm{marked}}$, take a marking η such that $\eta(H) = h$. Then $\eta^{-1}(\kappa)$ is of type $(1,1)$ and thus algebraic. The class H is contained in the wall $\eta^{-1}(\kappa)^\perp$ and thus not ample by the description of the Kähler cone, a contradiction.

Conversely, we consider a point $[x] \in \Omega_h$ not belonging to any Heegner divisor $\mathcal{H}_\kappa$. By Proposition 2.4, we know that $\mathcal{M}_h^{\mathrm{amp}}$ can be identified as a dense open subset of Ω_h by the marked period map $\wp$. If $[x]$ lies in this subset, then we know that $[x]$ is the period for some marked pair $(X,\eta) \in \mathcal{M}_h^{\mathrm{amp}}$ for which $\eta^{-1}(h)$ is ample; otherwise, since nefness is a closed condition, we can choose (X,η) so that $\eta^{-1}(h)$ is strictly nef, that is, it lies on the boundary of the Kähler cone $\mathcal{K}_X$. Since Kawamata–Morrison conjecture holds for X, we may conclude that $\eta^{-1}(h)$ lies on a hyperplane $D^\perp$ for some wall divisor $D \coloneqq \eta^{-1}(\kappa)$. But this means that the period $[x]$ is contained in the Heegner divisor $\mathcal{H}_\kappa$, where the wall class κ is orthogonal to h, and this is not the case by assumption. □

Finally we give a criterion for the existence of a wall class κ in $h^\perp$ for $\mathrm{K3}^{[m]}$-type and Kum_m-type.

Proposition 4.8. *For* $\mathrm{K3}^{[m]}$*-type or* Kum_m*-type, let* $h \in \Lambda$ *be an element of divisibility* γ*. Let* k *and* l *be integers satisfying the condition* (4)*. Then there is a wall divisor* $\kappa \in h^\perp$ *with* $\kappa^2 = 2l$ *and* $\kappa_* = k\delta_*$ *if and only if we have* $\gamma \mid k$*. Equivalently, this is the condition* $\operatorname{div} h \cdot \operatorname{div} \kappa \mid 2\widetilde{m}$*.*

Proof. Recall that $\Lambda = \Lambda_0 \oplus \mathbf{Z}\delta$. Write $h = \gamma a x + b\delta$ with $x \in \Lambda_0$ primitive, $\gcd(\gamma a, 2\widetilde{m}) = \gamma$, and $\gcd(\gamma a, b) = 1$. Write

$$\kappa = \frac{2\widetilde{m}}{d}(y + e\delta) + \frac{k}{d}\delta,$$

with $y \in \Lambda_0$. Thus κ being orthogonal to h is equivalent to

$$\gamma a(x, y) = b(2\widetilde{m}e + k).$$

Since $\gcd(\gamma, b) = 1$ and $\gamma \mid 2\widetilde{m}$, the condition $\gamma \mid k$ is clearly necessary. Conversely, if this condition is met, we show that there exist a suitable vector y and an integer e that give the desired κ. Since $\gcd(\gamma a, 2\widetilde{m}) = \gamma$, we may choose e such that

$$a \,\Big|\, \frac{2\widetilde{m}}{\gamma}e + \frac{k}{\gamma}\,.$$

Thus we only need to find $y \in \Lambda_0$ with required y^2 and (x, y). By Eichler's criterion, this can be done by taking $\phi \in \mathrm{O}(\Lambda_0)$ such that $\phi(x) = u_1' + \frac{x^2}{2}u_2'$ and then choosing y such that $\phi(y) = (x, y)u_2' + u_1'' + \frac{y^2}{2}u_2''$, where $\langle u_1', u_2'\rangle$ and $\langle u_1'', u_2''\rangle$ are two copies of hyperbolic plane U in Λ_0. □

In the proof, since we have explicitly described the classes h and κ, if we look at the sublattice $\langle h, \kappa, v\rangle$ in $\widetilde{\Lambda}$, its saturation is generated by the three classes $\frac{h - bv}{\gamma}$, $s = \frac{d\kappa - kv}{2\widetilde{m}}$, and v. So for this particular choice of κ, the discriminant of the saturation is $\left|\frac{2d^2nl}{\gamma^2\widetilde{m}}\right|$, while in general the discriminant would be this number divided by some square. Since the Mukai lattice $\widetilde{\Lambda}$ is unimodular, this is also the discriminant of the orthogonal $\langle h, \kappa, v\rangle^\perp$, which can be identified with the orthogonal $\langle h, \kappa\rangle^\perp$ in Λ. The latter is called the *transcendental lattice* of the Heegner divisor $\mathcal{H}_\kappa$. Its discriminant is also referred to as the discriminant of the Heegner divisor $\mathcal{H}_\kappa$. Therefore we have the following corollary.

Corollary 4.9. *Let* T *be a polarization type of square* $2n$ *and divisibility* γ *on hyperkähler manifolds of* $\mathrm{K3}^{[m]}$*-type or* Kum_m*-type. Let* k *and* l *be integers satisfying the condition* (4) *(which only depends on* m*) such that* $\gamma \mid k$*. For each connected component* $\mathcal{M}_\tau$ *of* $\mathcal{M}_T$*, the period map* $\wp_\tau$ *avoids at least one irreducible Heegner divisor* $\mathcal{H}_\kappa$ *of discriminant* $\left|\frac{2d^2nl}{\gamma^2\widetilde{m}}\right|$ *in* $\mathcal{P}_\tau$*, where* $d = \gcd(2\widetilde{m}, k)$*.*

For example, for $\mathrm{K3}^{[2]}$-type, we have already seen that (k, l) can be $(0, -1)$, $(1, -5)$, and $(1, -1)$. For a polarization type T of square $2n$, if the divisibility γ is equal to 2, the only possible case is $(0, -1)$ and we get a Heegner divisor of discriminant $2n$; if the divisibility γ is equal to 1, the three cases are all present and we get Heegner divisors of discriminant $8n$, $10n$, and $2n$. This result is however not exhaustive, since the sublattice we used above to compute the discriminant might still not be primitive in general, and the discriminant will be divided by some square. For example, when $\gamma = 1$, by [DM19, Theorem 6.1] it is also possible to have a Heegner divisor of discriminant $2n/5$ in the complement. Note also that there might be several irreducible Heegner divisors with the same discriminant while we have only obtained one of them.

Another simple example works for almost every polarization type T:

- If $\gamma \le \widetilde{m}$ we may take (k, l) to be $(\gamma, -\widetilde{m})$ (and $c = 0$), so the discriminant is equal to $2n$. In other words, for such a polarization type T, the restriction of the period map to every connected component of $\mathcal{M}_T$ will avoid an irreducible Heegner divisor of discriminant $2n$ in the period domain
- For a polarization type T not satisfying $\gamma \le \widetilde{m}$, γ is necessarily equal to the maximal value $2\widetilde{m}$. For $\mathrm{K3}^{[m]}$-type, we may take (k, l) to be $(0, -1)$ (so $c = -1$), and the discriminant is then equal to $\frac{2n}{m-1}$, so we get a similar conclusion. On the other hand, for Kum_m-type, a polarization type T of maximal divisibility $2\widetilde{m} = 2(m + 1)$ admits no orthogonal wall divisor: since by Proposition 4.8 we must have $k = 0$, so there exists no (k, l) satisfying the condition (4).

5. Two examples

Using the numerical condition (4), we can now compare the images by the period map of various components. Recall the picture of the polarized period map from (3). We prove the following result for $\mathrm{K3}^{[m]}$-type. Clearly the same idea can be adapted to Kum_m-type.

Proposition 5.1. *Let a be a positive integer.*

(i) *For hyperkähler manifolds of* $\mathrm{K3}^{[144^a+1]}$*-type, there is a unique polarization type T of square 288 and divisibility 12, for which the polarized moduli space $\mathcal{M}_T$ has exactly two components, with different images in $\mathcal{P}_T$ under the period map.*

(ii) *For hyperkähler manifolds of* $\mathrm{K3}^{[6^a+1]}$*-type, there is a unique polarization type T of square 2 and divisibility 1, for which the polarized moduli space $\mathcal{M}_T$ is connected. The group G is isomorphic to $\mathbf{Z}/2\mathbf{Z}$, and the image of the period map in $\mathcal{P}_\tau$ is not G-invariant above $\mathcal{P}_T$.*

Proof. For (i), we may check by Proposition 3.5 that such polarization type is unique and the polarized moduli space $\mathcal{M}_T$ has exactly two components. Note that by Proposition 3.4, $\gamma = 12$ is the smallest divisibility for the moduli space $\mathcal{M}_T$ to have more than one component.

As $D(\Lambda) = \mathbf{Z}/(2 \cdot 144^a)\mathbf{Z}$ and $\rho(2 \cdot 144^a) = 2$, by Lemma 3.2 we have $\mathrm{O}(D(\Lambda)) = \{\pm 1, \pm g\}$. For $h \in T$, the class h_* is of order 12 in $D(\Lambda)$. So for any $\phi \in \mathrm{O}(\Lambda, h)$, we have $\chi(\phi) = 1$ since 1 is the unique element in $\mathrm{O}(D(\Lambda))$ that is $\equiv 1 \pmod{12}$. This shows that $\mathrm{O}(\Lambda, h) \subset \widetilde{\mathrm{O}}(\Lambda)$ and consequently, the group $\mathrm{O}^+(\Lambda, h)/\operatorname{Mon}(\Lambda, h)$ is trivial. In this case, both period domains $\mathcal{P}_\tau$ are canonically isomorphic to $\mathcal{P}_T$.

Since $\mathcal{M}_T$ has two components, we may choose $h, h' \in T$ belonging to different $\operatorname{Mon}(\Lambda)$-orbits or equivalently, $\widehat{\mathrm{O}}(\Lambda)$-orbits, as we have seen in the proof of Proposition 3.4 that they are the same. There exists $\psi \in \mathrm{O}(\Lambda) \smallsetminus \widehat{\mathrm{O}}(\Lambda)$ such that $\psi(h) = h'$. We may assume that $\chi(\psi) = g$. Consider the period domain $\mathcal{P}_T$ realized as the quotient $\Omega_h/\mathrm{O}^+(\Lambda, h)$ or $\Omega_{h'}/\mathrm{O}^+(\Lambda, h')$. The automorphism ψ induces an identification between the two, which maps each Heegner divisor $\mathcal{H}_\kappa$ to $\mathcal{H}_{\psi(\kappa)}$.

We consider a wall class $\kappa \in h^\perp$ with square $2l$ and $\kappa_* = k\delta_*$. The class $\kappa' = \psi(\kappa)$ has the same square $2l$ while $\kappa'_* = k'\delta_*$ with $k' \equiv gk \pmod{2 \cdot 144^a}$. For κ' to also define a wall class, we need

$$c' = c + \frac{k'^2 - k^2}{4 \cdot 144^a} \geq -1$$

to hold. So the idea is to choose some suitable k, l for which this condition fails. We let $k = 12g_0$ such that $k \equiv 12g \pmod{2 \cdot 144^a}$ (so g_0 is the residue of g modulo $24 \cdot 144^{a-1}$). Since $g \neq \pm 1$ in $\mathrm{O}(D(\Lambda))$, g_0 cannot be ± 1 hence we have $g_0^2 > 1$. Then we can let $c = -1$ and find the value for l using (4). By Proposition 4.8, there exists indeed such a wall class $\kappa \in h^\perp$. On the other hand, the choice of k means $k' = 12$, so $c' = -1 + \frac{12^2 - 12^2 g_0^2}{4 \cdot 144^a} < -1$, and κ' is not a wall class. This shows that the same Heegner divisor inside $\mathcal{P}_T$ is avoided by the period map for one component but not for the other. Thus their images in $\mathcal{P}_T$ by the period map are not the same.

For (ii), once again we may verify by Proposition 3.5 that there is a unique such polarization type T with one connected component. And by Lemma 3.2, since $D(\Lambda) = \mathbf{Z}/(2 \cdot 6^a)\mathbf{Z}$ and $\rho(2 \cdot 6^a) = 2$, we have $\mathrm{O}(D(\Lambda)) = \{\pm 1, \pm g\}$.

Since this $\mathrm{O}(\Lambda)$-orbit is unique, we may take $h = u_1' + u_2'$, where $\langle u_1', u_2' \rangle$ is a copy of U. The group $\mathrm{O}(\Lambda, h)$ contains $\mathrm{O}(\Lambda, U) \coloneqq \{\phi \in \mathrm{O}(\Lambda) \mid \phi|_U = \mathrm{Id}\}$ as a subgroup, which is isomorphic to $\mathrm{O}(U^\perp)$ since U is a direct summand. Moreover, the inclusion $\mathrm{O}(U^\perp) \simeq \mathrm{O}(\Lambda, U) \hookrightarrow \mathrm{O}(\Lambda)$ induces an isometry between the two discriminant groups. We use Proposition 2.12 on $\mathrm{O}(U^\perp)$ to deduce that the homomorphism $\chi\colon \mathrm{O}(\Lambda) \to \mathrm{O}(D(\Lambda))$ when restricted to $\mathrm{O}(\Lambda, U)$, is still surjective. In particular, there is $\phi \in \mathrm{O}(\Lambda, h)$ such that $\chi(\phi) = g$. On the other hand, following the proof of Proposition 2.15, there is an element $R \in \mathrm{O}(\Lambda, h)$ such that $\sigma(R) = -1$ and $\chi(R) = 1$. Let ψ be ϕ if $\sigma(\phi)$, and $R \circ \phi$ otherwise. Then ψ is in $\mathrm{O}^+(\Lambda, h)$ with $\chi(\psi) = g$. Consequently, the group $\mathrm{O}^+(\Lambda, h)/\operatorname{Mon}(\Lambda, h)$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$.

As in the previous case, we consider a wall class $\kappa \in h^\perp$ with square $2l$ and $\kappa_* = k\delta_*$, for $k = g$ and $c = -1$. Such a class exists by Proposition 4.8. However, the class $\kappa' = \psi(\kappa)$ will have $k' = 1$, so $c' = -1 + \frac{1^2 - g^2}{4 \cdot 6^a} < -1$ and κ' is not a wall class. This shows that there are two Heegner divisors in $\mathcal{P}_\tau$ that can be mapped to each other under the action of $\mathrm{O}^+(\Lambda, h)/\operatorname{Mon}(\Lambda, h)$, but one is avoided by the period map and the other is not. Thus we see in particular that the group G is non-trivial and therefore also isomorphic to $\mathbf{Z}/2\mathbf{Z}$, and the image of the period map is not G-invariant. □

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